# Existence, uniqueness and numerical investigation of segregation models

UT Austin | Portugal
INTERNATIONAL COLLABORATORY FOR EMERGING TECHNOLOGIES, COLAB

Farid Bozorgnia
Técnico Lisboa

FCT Fundação para a Ciência e a Tecnologia

## Adjacent segregation:

Particles annihilate or interact on contact line or common surface of separation. Appears in competition models of Lotka-Volterra type and Variational problems.

**Problem (A):** Let $\Omega \subset \mathbb{R}^d$ be a connected, bounded domain with smooth boundary and $m$ be a fixed integer. The density of i-th component $u_i(x) : i = 1, \cdots, m$ with the internal dynamic is prescribed by $f_i$.

The steady-states of $m$ competing components in $\Omega$ is given by

$$\begin{cases} -\Delta u_i^\varepsilon = -\frac{1}{\varepsilon} u_i^\varepsilon(x) \sum_{j\neq i}^m a_{ij}\, u_j^\varepsilon(x) + f_i(x, u_i^\varepsilon(x)) & \text{in } \Omega \\ u_i \geq o & \text{in } \Omega \\ u_i(x) = \phi_i(x) & \text{on } \partial\Omega. \end{cases} \quad (1)$$

The boundary values $\phi_i$ are non-negative and have disjoint supports on the boundary, i.e,

$$\phi_i \cdot \phi_j = 0 \quad \text{on} \quad \partial\Omega.$$

We call the $m$-tuple $U = (u_1, \cdots, u_m) \in (H^1(\Omega))^m$, *segregated states* if

$$u_i(x) \cdot u_j(x) = 0, \text{ a.e. for } \quad i \neq j, \ x \in \Omega.$$

**Problem (B):** Consider the following minimization problem

$$\text{Minimize } E(u_1, \cdots, u_m) = \int_\Omega \sum_{i=1}^m \left( \frac{1}{2} |\nabla u_i|^2 + f_i u_i \right) dx, \quad (2)$$

over the set

$$K = \{(u_1, \ldots, u_m) \in (H^1(\Omega))^m : u_i \geq 0, u_i \cdot u_j = 0, \text{ in } \Omega, \ u_i = \phi_i \text{ on } \partial\Omega\}.$$

Here $\phi_i \in H^{\frac{1}{2}}(\partial\Omega)$ with property $\phi_i \cdot \phi_j = 0$, $\phi_i \geq 0$ on the boundary $\partial\Omega$. Also we assume that $f_i$ is uniformly continuous and $f_i(x) \geq 0$.

**Theorem** ( see [3]):
Let $U^\varepsilon = (u_1^\varepsilon, \ldots, u_m^\varepsilon)$ be a solution of system at fixed $\varepsilon$. Let $\varepsilon \to 0$, then there exists $U \in (H^1(\Omega))^m$ such that for all $i = 1, \cdots, m$:

1. up to a subsequences, $u_i^\varepsilon \to u_i$ strongly in $H^1(\Omega)$,
2. $u_i \cdot u_j = 0$ if $i \neq j$ a.e in $\Omega$,
3. $\Delta u_i = 0$ in the set $\{u_i > 0\}$.
4. Let $x$ belongs to interface such that $m(x) = 2$ then $\lim_{y \to x} \nabla u_i(y) = -\lim_{y \to x} \nabla u_j(y)$.

The similar results for Problem (B) holds with difference $\Delta u_i = f_i$ in the set $\{u_i > 0\}$.

## Numerical approximation for System as $\varepsilon \to 0$:

The algorithm for an arbitrary $m$ is as follows. Suppose there is a grid on the domain $\Omega$, then the second method can be formulated as

- **Initialization: for** $l = 1, \ldots, m$

$$u_l^0(x_i, y_j) = \begin{cases} 0 & \text{if } (x_i, y_j) \text{ is an interior point,} \\ \phi_l(x_i, y_j) & \text{if } (x_i, y_j) \text{ is a boundary point.} \end{cases}$$

- **Step** $k+1$, $k \geq 0$:
  Let $\overline{u_l}(x_i, y_j)$ denote the average of $u_l$ for all neighbors of the point$(x_i, y_j)$.
  We iterate for all interior points by

$$u_l^{(k+1)}(x_i, y_j) = \max\left( \overline{u_l}^{(k)}(x_i, y_j) - \sum_{p \neq l} \overline{u_p}^{(k)}(x_i, y_j), 0 \right), \ l = 1, \ldots m. \quad (3)$$

## Segregation at Distance:

In this model, components interact at a distance from each other, [2]:

$$\begin{cases} -\Delta u_i^\varepsilon = -\frac{1}{\varepsilon} u_i^\varepsilon \sum_{j \neq i} H(u_j^\varepsilon)(x) & \text{in } \Omega, \\ u_i(x) = \phi_i(x) & \text{in } (\partial\Omega)_1, \\ i = 1, \cdots, m. \end{cases} \quad (4)$$

where

$$H(u_j^\varepsilon)(x) = \int_{B_1(x)} u_j^\varepsilon(y) dy, \quad \text{or} \quad H(u_j^\varepsilon)(x) = \sup_{y \in B_1(x)} u_j^\varepsilon(y).$$

In (4), $(\partial\Omega)_1 := \{x \in \Omega^c : dist(x, \Omega) \leq 1\}$. Assumptions: $\phi_i(x)$ are non-negative $C^1$ functions, have disjoint supports in distance more than one:

$$(supp\, \phi_i(x))_1 \cap (supp\, \phi_j(x))_1 = \emptyset.$$

**Theorem**(see [1]):
For each $\varepsilon > 0$, there exist a unique positive solution $(u_1^\epsilon, \cdots, u_m^\epsilon)$ of systems in (1) and (4).

Remark: The proof is constructive can be used numerical approximation.

- Consider the harmonic extension $u_i^0$ for $i = 1, \cdots, m$ given by

$$\begin{cases} -\Delta u_i^0 = 0 & \text{in } \Omega, \\ u_i^0 = \phi_i & \text{on } \partial\Omega, \end{cases}$$

- Given $u_i^k$, consider the solution of the following linear system

$$\begin{cases} \Delta u_i^{k+1} = \frac{1}{\varepsilon} u_i^{k+1} \sum_{i \neq j} H(u_j^k)(x) & \text{in } \Omega, \\ u_i^{k+1}(x) = \phi_i(x) & \text{on } (\partial\Omega)_1, \end{cases}$$

### Monotone property of scheme

$$u_i^0 \geq u_i^2 \geq \cdots \geq u_i^{2k} \geq \cdots \geq u_i^{2k+1} \geq \cdots \geq u_i^3 \geq u_i^1, \quad \text{in } \Omega,$$

which implies

$$u_i^{2k} \to u_i^\star \quad \text{and} \quad u_i^{2k+1} \to u_i^\diamond \quad \text{uniformly in } \Omega.$$

Therefore $u_i^\star \geq u_i^\diamond$. Next step is to show $u_i^\star = u_i^\diamond$.

Assume there exist another solution $(w_1, \cdots, w_n)$ then:

$$u_i^{2k+1} \leq w_i \leq u_i^{2k}, \quad \text{which shows } u_i = w_i. \quad (5)$$

### Theorem

*Let $u_i$, $i = 1, \cdots, m$ be the limiting solutions as $\varepsilon$ tends to zero in (4). Then*

- *The function $u_i$ is is harmonic in it's support and Lipschitz continuous in domain $\Omega$.*
- *The free boundaries $\Gamma_i = \partial\{x \in \Omega : u_i(x) > 0\}$, $\Gamma_j = \partial\{x \in \Omega : u_j x) > 0\}$, have distance one from each other.*

## RNA interactions between Ribonucleic acid (RNA)

- Ribonucleic acid (RNA): class of important biological molecules that play crucial roles in coding, decoding, regulation, and expression of genes.
- Messenger RNA (mRNA): The type of RNA that carries information from DNA to the ribosome, the sites of protein synthesis in the cell.
- Small, non-coding RNAs (sRNA): regulate events such as cell growth and tissue differentiation through binding and reacting with mRNA in a cell.

Reaction- Diffusion System for RNA Interactions

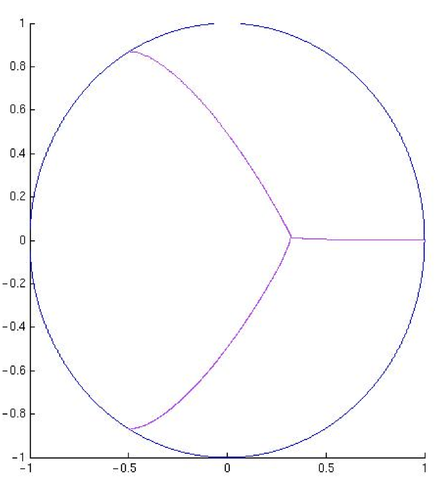
Consider the steady state of following system

$$\begin{cases} \frac{\partial u_i}{\partial t} = \Delta u_i - \beta_i u_i - k_i u_i v + \alpha_i & \text{in } \Omega \times (0, \infty), \\ \frac{\partial v}{\partial t} = \Delta v - \beta v - \sum_{i=1}^m k_i u_i v + \alpha & \text{in } \Omega \times (0, \infty), \\ u_i(\cdot, 0) = u_{i\,0}, \quad v(\cdot, 0) = v_0 & \text{in } \Omega, \\ \frac{\partial u_i}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases} \quad (6)$$

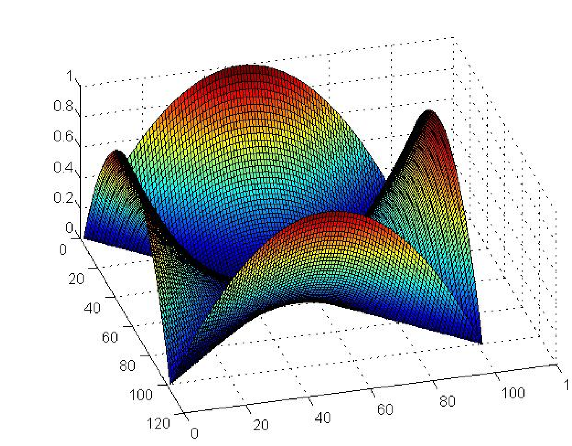
$u_{i\,0}$ $(i = 1, \ldots, m)$ and $v_0$ are nonnegative functions in $\Omega$.

## Examples

1. In system (1), let $\Omega = B_1, m = 3$. The boundary values $\phi_i$ for $i = 1, 2, 3$ are $\phi_1(1, \Theta) = |\sin(\frac{3}{2}\Theta)|$, $\phi_2(1, \Theta) = |\sin(\frac{3}{2}\Theta)|$, $\phi_3(1, \Theta) = 4|\sin(\frac{3}{2}\Theta)|$. The Figure below shows the free boundaries for limiting problem.

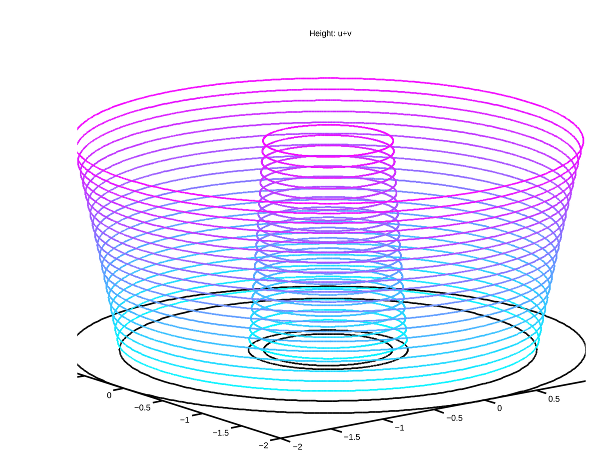


2. $\Omega = [0, 1] \times [0, 1]$ ,$\phi_1 = 1 - x^2, \phi_2 = 1 - y^2, \phi_3 = 1 - x^2, \phi_4 = 1 - y^2$. The picture depicts $u_1 + u_2 + u_3 + u_4$.



3. Consider system (4) with $m = 2, \Omega = B_2 \setminus B_{.5}$. The boundary values are

$$u = 1 \quad \text{on } \partial B_{.5} \quad v = 1 \quad \text{on } \partial B_2,$$



## References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[2] L. Caffarelli, S. Patrizi, and V. Quitalo, *On a long range segregation model.* http://arxiv.org/abs/1505.05433.

[3] M. Conti, S. Terracini, and G. Verzini, *Asymptotic estimate for spatial segregation of competitive systems.* Advances in Mathematics. **195,** 524-560, (2005).